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Scattering of Klein–Gordon and Maxwell Waves by an Ellis Geometry

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The problem of scattering of scalar and electromagnetic waves by an Ellis geometry, with the two asymptotically flat regions observationally identified, is formulated and solved. The results are consistent with the interpretation of the Ellis geometry as an extended particle.

1. INTRODUCTION

While the coupled classical Einstein–Maxwell equations do not admit regular static solutions, it is known (Clément, 1981) that the coupled Einstein–Maxwell–Higgs (EMH) equations, derived from the action

$$S = \int d^4x |g|^{1/2} \left[-\frac{1}{2\kappa} g^{\mu\nu} R_{\mu\nu} - \frac{1}{4} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} + \frac{\varepsilon}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{\varepsilon}{2} \frac{\mu^2}{\kappa f^2} \left(1 - \frac{\kappa f^2}{2} \phi^2 \right)^2 \right] \tag{1}$$

admit regular localized static solutions, in the case of a repulsive Higgs field ($\varepsilon < 0$). These solutions, parametrized by a characteristic length r_0, are given in the Prasad–Sommerfield limit $\mu^2 \to 0$ (vanishing self-coupling of the

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Higgs field) by (Clément, 1981, Bronnikov, 1973)

$$ds^2 = g_{00}\,dt^2 - \frac{1}{g_{00}}\left(1+\frac{r_0^2}{r^2}\right)^2 d\mathbf{x}^2$$

$$A_\mu\,dx^\mu = \pm\left(\frac{2}{\kappa}\right)^{1/2}\cos\left(\frac{\pi\lambda}{2}\right)\tan(\lambda\eta)\,dt$$

$$\phi = \left(\frac{2}{\kappa}\right)^{1/2}\frac{2}{\pi f}\eta \tag{2}$$

where $\lambda = (1-4/\pi^2 f^2)^{1/2}$, and

$$g_{00} = \frac{\cos^2(\pi\lambda/2)}{\cos^2(\lambda\eta)}$$

$$\eta = 2\arctan\left(\frac{r}{r_0}\right) - \frac{\pi}{2} \tag{3}$$

The non-Newtonian, purely spatial part of the geometry (2), first studied by Ellis (1973), describes a space consisting of two asymptotically flat regions ($r\to\infty, r\to 0$) smoothly connected by a neck ($r = r_0$). This peculiar topology is responsible for the existence of a static solution to the Laplace equation for the electric potential A_0. The asymptotic behaviors of the gravitational and electromagnetic potentials (2) are compatible with the interpretation (Clément, 1981) of such a solution as an extended particle ("gravitational soliton") of mass

$$M = \frac{2}{\kappa}8\pi\lambda r_0 \tan\left(\frac{\pi\lambda}{2}\right) \tag{4}$$

and charge

$$Q = \pm\left(\frac{2}{\kappa}\right)^{1/2}\frac{8\pi\lambda r_0}{\cos(\pi\lambda/2)} \tag{5}$$

Actually, Q is the charge defined as the electric flux through a sphere of radius $R\to\infty$. If we compute instead the electric flux through a sphere of radius $R\to 0$ (the other asymptotic region), we obtain the result $-Q$. It follows that the coordinate inversion

$$\mathbf{x} \to \frac{r_0^2}{r^2}\mathbf{x}$$

which maps space onto itself (each point in the region $r > r_0$ going over to the symmetrical point in the region $r < r_0$ and vice versa) is to be interpreted as charge conjugation.

Such an interpretation may be consistent with macroscopic observation if we make the (admittedly speculative) assumption that any system of N elementary particles (including measuring apparatus and observers) are localized solutions of the EMH equations (multisolitons) for which "space" (timelike slices of the 4-geometry) consists of two asymptotically flat, symmetrical regions connected by N necks, each neck corresponding to a particle. The neck structure can only be felt at the microscopic level (in high-energy experiments), while at the macroscopic level we recover the familiar description of a system of N quasi-point particles in a two-sheeted Euclidean space, the two sheets of which are related by charge conjugation.

The particles of our model universe interact through three forces: two (gravitational and electromagnetic) long range, and a short-range (in the physical case $\mu^2 > 0$) force mediated by the scalar field ϕ. A first step in the study of the resulting dynamics is the investigation of the interaction of an isolated gravitational soliton with a gravitational, electromagnetic, or scalar wave, considered as a disturbance of the localized solution (2). Assuming that gravitational interactions of elementary particles are negligible at currently available energies, we shall restrict ourselves to the study of the scattering of an electromagnetic or scalar disturbance by the gravitational soliton, neglecting the back-reaction of the energy-momentum density carried by the wave on the metric. This amounts to solving the uncoupled Maxwell or Klein–Gordon (in the case $\mu^2 = 0$) equations, with suitable boundary conditions, in a space-time of metric (2). Furthermore, noting that the dimensionless constant λ is of the order of the square root of the ratio of the gravitational radius to the electromagnetic radius of the gravitational soliton, and thus would be very small for elementary particles ($\lambda \sim 2 \times 10^{-18}$ for a proton), we shall approximate g_{00} by 1, thus neglecting gravitational scattering proper before scattering by the spatial Ellis geometry.

Such a reduction of dynamics to geometry may be a dangerous oversimplification, as will turn out when we compute the cross section for the scattering of light by the Ellis geometry, and find a vanishing result in the low-energy limit, instead of the expected finite Thomson result. This probably means that we should not neglect back-reaction, but treat the full geometrodynamical problem, by linearizing the coupled field equations off the solution (2), along the lines of Moncrief (1975). In the meantime, a sufficient motivation for the present work is that an understanding of geometrical scattering by a gravitational soliton, even if it is not the main effect, is a necessary step toward the computation of the side effects due to the finite extension of the charge.

The study of the scattering of a massless Klein–Gordon wave or a Maxwell wave by the Ellis geometry is reduced, in Section 2 of this paper, to a quantum mechanical problem in one space dimension. The corresponding phase shifts are computed in Section 3 and their low-energy behavior is discussed in Section 4.

2. RADIAL EQUATIONS

The Ellis geometry, first derived as a solution to the Einstein–Klein–Gordon equations (for a massless repulsive scalar field) (Ellis, 1973; Clément, 1979) is obtained from the self-consistent static solution (2) by the limit $g_{00} \to 1$:

$$ds^2 = dt^2 - \left(1 + \frac{r_0^2}{r^2}\right)^2 d\mathbf{x}^2 \qquad (6)$$

For small λ, equations (4) and (5) give the characteristic radius r_0 in terms of the mass and charge of the extended particle as

$$r_0 = \frac{Q^2}{16M} \qquad (7)$$

which is of the order of the classical electromagnetic radius of a charged particle.

For practical purposes it is convenient to describe the Ellis geometry in terms of radial-distance preserving spherical coordinates which are the usual polar angles θ, φ, and the proper radial coordinate u defined by

$$du = \left(1 + \frac{r_0^2}{r^2}\right) dr \qquad (8)$$

or

$$u = r - \frac{r_0^2}{r} \qquad (9)$$

This gives

$$ds^2 = dt^2 - du^2 - \left(\rho_0^2 + u^2\right)\left(d\theta^2 + \sin^2\theta \, d\varphi^2\right) \qquad (10)$$

where

$$\rho_0 = 2r_0 \qquad (11)$$

Thus, $|u|$ is the proper radial distance to the neck $u=0$, which is a two-sphere of area $4\pi\rho_0^2$. The coordinate u is positive on one side of the neck, negative on the other, the two asymptotically flat regions of the geometry corresponding to $u \to \pm\infty$. Thus, charge conjugation is implemented by the transformation $u \to -u$.

The stationary massless Klein–Gordon equation for the positive-frequency component ψ of ϕ,

$$\Delta\psi + \omega^2\psi = 0 \tag{12}$$

may be written, in proper spherical coordinates, as

$$\frac{1}{\rho_0^2+u^2}\left[\frac{\partial}{\partial u}\left(\left(\rho_0^2+u^2\right)\frac{\partial\psi}{\partial u}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}\right]+\omega^2\psi=0 \tag{13}$$

The expansion of ψ in spherical harmonics,

$$\psi(\mathbf{x}) = \sum_{l=0}^{\infty} \psi_l(u) Y_l^m(\theta,\varphi) \tag{14}$$

reduces (13) to the radial equation

$$f_l''(u)+\left[\omega^2-\frac{l(l+1)}{\rho_0^2+u^2}-\frac{\rho_0^2}{\left(\rho_0^2+u^2\right)^2}\right]f_l(u)=0 \tag{15}$$

where

$$f_l(u) = \left(\rho_0^2+u^2\right)^{1/2}\psi_l(u) \tag{16}$$

The similar reduction of the stationary Maxwell equations to a single radial equation is achieved by the vectorial method of Stratton (1941), which we outline for a general static spherically symmetric geometry. Choosing spatially isotropic coordinates such that

$$ds^2 = e^{\alpha(r)-\beta(r)}\,dt^2 - e^{\alpha(r)+\beta(r)}\,d\mathbf{x}^2 \tag{17}$$

the stationary Maxwell equations for the positive frequency components E_+^i and B_+^i of the electric and magnetic fields

$$\begin{aligned} E^i &\equiv F^{i0} \\ B^i &\equiv -\frac{1}{2|g|^{1/2}}\varepsilon^{ijk}F_{jk} \end{aligned} \tag{18}$$

may be written as

$$\nabla_x e^{2\alpha}\mathbf{E}_+ = i\omega e^{2\alpha+\beta}\mathbf{B}_+$$

$$\nabla_x e^{2\alpha}\mathbf{B}_+ = -i\omega e^{2\alpha+\beta}\mathbf{E}_+ \tag{19}$$

Let $\psi(\mathbf{x})$ be a scalar field, and define two vector fields $\mathbf{M}(\mathbf{x})$ and $\mathbf{N}(\mathbf{x})$ such that

$$e^{3\alpha}\mathbf{M} = \nabla_x \mathbf{x} e^{\alpha+\beta}\psi$$

$$\nabla_x e^{2\alpha}\mathbf{M} = \omega e^{2\alpha+\beta}\mathbf{N} \tag{20}$$

We then obtain

$$\nabla_x e^{2\alpha}\mathbf{N} = \omega e^{2\alpha+\beta}\mathbf{M} \tag{21}$$

provided ψ is a solution of the equation

$$e^{-3\beta}\nabla\left(e^{\beta}\nabla\psi\right) + e^{-2\beta}r^{-1}\left(r\beta'\right)'\psi + \omega^2\psi = 0 \tag{22}$$

The general solution of the Maxwell equations (19) may then be expanded in terms of the linearly independent solutions ψ_n of equation (22):

$$\mathbf{E}_+ = \sum_n \left(a_n\mathbf{M}_n + b_n\mathbf{N}_n\right)$$

$$\mathbf{B}_+ = -i\sum_n \left(b_n\mathbf{M}_n + a_n\mathbf{N}_n\right). \tag{23}$$

In our case $g_{00} = 1$, $\beta = \alpha$, and the scalar wave equation (22) may be written

$$\Delta\psi + \tfrac{1}{2}R^{\rho}_{\rho}\psi + \omega^2\psi = 0 \tag{24}$$

where

$$R^{\rho}_{\rho} \equiv R_{ij}\frac{x^i x^j}{r^2} \tag{25}$$

is the radial component of the Ricci tensor (equal to the scalar curvature R in the case of the Ellis geometry). The separation of this equation in proper spherical coordinates may be carried out as in the case of the Klein–Gordon equation, leading to the radial equation

$$f_l''(u)+\left[\omega^2-\frac{l(l+1)}{\rho_0^2+u^2}\right]f_l(u)=0 \tag{26}$$

In order to solve the radial equations (15) or (26), we must add suitable boundary conditions. For similar quantum mechanical equations in Euclidean space, we would have a boundary condition at infinity, and a boundary condition [$f_l(0)=0$] at the origin $r=0$ coming from the fact that the radial variable r is constrained to be nonnegative. In our case the radial variable u is unconstrained so that there is *a priori* no reason for a boundary condition at $u=0$; however, there are two points at infinity, and so we have again two boundary conditions.

Actually, the two boundary conditions for $u\to\pm\infty$ may be replaced, as in the Euclidean case, by two boundary conditions at $u\to+\infty$ and $u=0$, if we take into account the properties of the scalar or electromagnetic fields under charge conjugation. The electromagnetic field is, by definition, odd under charge conjugation. The static Klein–Gordon field ϕ given by (2) is also odd under charge conjugation, and the same must be true for its perturbations. The corresponding radial functions must therefore be odd in u:

$$f_l(-u)=-f_l(u) \tag{27}$$

The radial equations (15) and (26) being even in u, conditions (27) are equivalent to the regularity conditions:

$$f_l(0)=0 \tag{28}$$

Because scattering has to do only with asymptotic measurements, the flat-space partial-wave formalism for wave scattering (Newton, 1966) applies here as well. The scattering amplitude $F(\Omega)$ is defined, in the case of a massless scalar wave, by the asymptotic decomposition

$$\phi(\mathbf{x}) \underset{\substack{\mathbf{x}\to\infty\\(u>0)}}{\sim} e^{i\mathbf{k}\cdot\mathbf{x}}+F(\Omega)\frac{e^{i\omega u}}{u} \qquad (|\mathbf{k}|=\omega) \tag{29}$$

and may be obtained by solving the radial equations (15) with the boundary conditions (28), and computing the phase shifts δ_l from the asymptotic behavior

$$f_l(u) \underset{u\to+\infty}{\sim} A_l\sin\left(\omega u-\frac{l\pi}{2}+\delta_l\right) \tag{30}$$

The scattering amplitude is then given by

$$F(\Omega)=\sum_{l=0}^{\infty}\frac{2l+1}{2i\omega}\left(e^{2i\delta_l}-1\right)P_l(\cos\theta) \tag{31}$$

and the scattering cross section is

$$\sigma=\int|F(\Omega)|^2\,d\Omega=\frac{4\pi}{\omega^2}\sum_{l=0}^{\infty}(2l+1)\sin^2\delta_l \tag{32}$$

In the case of an electromagnetic wave, the scattering amplitude $F(\Omega)$ is a matrix defined by the asymptotic decomposition of the electric field

$$\mathbf{E}_+(\mathbf{x})\underset{\substack{\mathbf{x}\to\infty\\(u>0)}}{\sim}\sum_{\lambda=1}^{2}\mathbf{E}_0^{(\lambda)}\left[\boldsymbol{\varepsilon}(\mathbf{k},\lambda)e^{i\mathbf{k}\cdot\mathbf{x}}+\sum_{\lambda'=1}^{2}F_\lambda^{\lambda'}(\mathbf{k},\mathbf{k}')\boldsymbol{\varepsilon}(\mathbf{k}',\lambda')\frac{e^{i\omega u}}{u}\right] \tag{33}$$

where the $\boldsymbol{\varepsilon}(\mathbf{k},\lambda)$ are two independent polarization vectors relative to the direction $\mathbf{k}$, and $\mathbf{k}'=\omega\mathbf{x}/u$. This matrix may be expanded on the basis of "electric" and "magnetic" vector spherical harmonics which are proportional to the vectors $\underset{0}{\mathbf{N}}{}_l^m$ and $\underset{0}{\mathbf{M}}{}_l^m$ defined by the Euclidean version ($\alpha=\beta=0$) of equations (20) applied to the scalar spherical harmonics [$\psi(\mathbf{x})=Y_l^m(\theta,\varphi)$]. The resulting partial-wave matrix amplitudes, diagonal in this basis, are proportional to

$$e^{2i\Delta_l}-1=\begin{pmatrix}e^{2i\delta_{l(e)}}-1 & 0\\ 0 & e^{2i\delta_{l(m)}}-1\end{pmatrix} \tag{34}$$

where the "electric" and "magnetic" phase shifts $\delta_{l(e)}$ and $\delta_{l(m)}$ may be computed from the asymptotic behaviors of $\mathbf{N}_l^m$ and $\mathbf{M}_l^m$.

In our case, the stationary Maxwell equations (19) are completely symmetric between the electric and magnetic fields, and so these two phase shifts are equal to the phase shifts obtained from the asymptotic behavior (30) of the solution of the radial electromagnetic equation (26). The total cross section (summed over final polarizations and averaged over initial polarizations) is then

$$\sigma=\frac{2\pi}{\omega^2}\sum_{l=1}^{\infty}(2l+1)\left(\sin^2\delta_{l(e)}+\sin^2\delta_{l(m)}\right)=\frac{4\pi}{\omega^2}\sum_{l=1}^{\infty}(2l+1)\sin^2\delta_l \tag{35}$$

3. PHASE SHIFTS

Radial equations (15) and (26) are two special cases of the ubiquitous spheroidal wave equation (Erdélyi, 1955a; Abramowitz and Stegun, 1965), which also occurs in the separation of the flat-space wave equation in spheroidal coordinates, as well as in the separation of the wave equation in the Kerr–Newman metric in spherical coordinates (Misner et al., 1973). The reduction of equations (15) and (26) to the standard form (Erdélyi, 1955a) of the spheroidal wave equation is achieved by putting

$$z = i\rho_0^{-1}u$$
$$\theta = -\frac{\omega^2\rho_0^2}{4}, \qquad \lambda = l(l+1) \tag{36}$$

and using (16); we thus obtain

$$(1-z^2)\psi_l''(z) - 2z\psi_l'(z) + \left[\lambda + 4\theta(1-z^2) - \mu^2(1-z^2)^{-1}\right]\psi_l(z) = 0 \tag{37}$$

where $\mu = 0$ for the Klein–Gordon radial equation (15), $\mu = \pm 1$ for the Maxwell radial equation (26).

The solutions of this equation have three singular points $z = \pm 1$ ($u = \pm i\rho_0$) and $z = \infty$ ($u = \infty$). We look for solutions which are analytic in a domain including the axis of real u (imaginary z); such a domain is the complex z plane with the cut$]-\infty, -1] \cup [+1, +\infty[$ (Figure 1). It is inter-

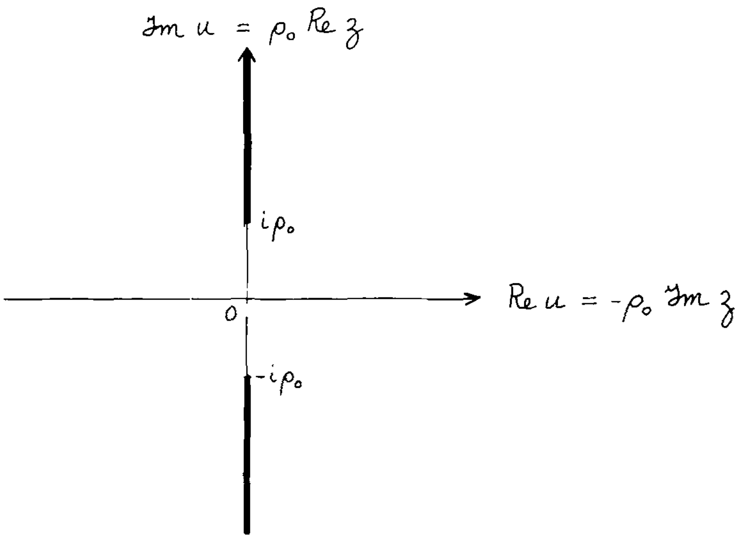

Fig. 1. The domain of analyticity of $\psi_l(u)$.

esting to note that this cut separates the complex u plane into two regions $\mathrm{Re}\,u>0$ and $\mathrm{Re}\,u<0$, connected by the segment $]-i\rho_0, +i\rho_0[$. In the "macroscopic" limit $\rho_0 \to 0$, the two regions $\mathrm{Re}\,u>0$ and $\mathrm{Re}\,u<0$ become disconnected, in accordance with our interpretation of the macroscopic limit of the Ellis geometry as a two-sheeted Euclidean space with a point singularity at the origin.

Two linearly independent solutions of the spheroidal wave equation with this cut are $\bar{P}s_\nu^\mu(z)$ and $\bar{Q}s_\nu^\mu(z)$, defined by the (analytically continued) series expansions in Legendre functions "on the cut" (Erdélyi, 1955a):

$$\begin{aligned} \bar{P}s_\nu^\mu(z) &= \sum_{r=-\infty}^{+\infty} (-1)^r a_{\nu,r}^\mu(\theta)\bar{P}_{\nu+2r}^\mu(z) \\ \bar{Q}s_\nu^\mu(z) &= \sum_{r=-\infty}^{+\infty} (-1)^r a_{\nu,r}^\mu(\theta)\bar{Q}_{\nu+2r}^\mu(z) \end{aligned} \tag{38}$$

where the real coefficients $a_{\nu,r}^\mu(\theta)$ may be determined recursively, starting from

$$a_{\nu,r}^\mu(0) = \delta_{r0}$$

and the characteristic exponent ν is given in terms of λ and θ by a transcendental equation, which may be expanded (Abramowitz and Stegun, 1965) in powers of θ as

$$\lambda = \nu(\nu+1) - 2\theta\frac{\nu(\nu+1)+\mu^2-1}{\nu(\nu+1)-3/4} + O(\theta^2) \tag{39}$$

(using Erdélyi's, and not Abramowitz's, definition of λ !).

To obtain the regular solution, we use the (analytically continued) relations (Erdélyi, 1955b):

$$\begin{aligned} \bar{P}_\nu^\mu(-z) &= \cos[(\mu+\nu)\pi]\,\bar{P}_\nu^\mu(z) - \frac{2}{\pi}\sin[(\mu+\nu)\pi]\,\bar{Q}_\nu^\mu(z) \\ \bar{Q}_\nu^\mu(-z) &= -\frac{\pi}{2}\sin[(\mu+\nu)\pi]\,\bar{P}_\nu^\mu(z) - \cos[(\mu+\nu)\pi]\,\bar{Q}_\nu^\mu(z) \end{aligned} \tag{40}$$

The expansions (38) show that these relations also hold for the functions $\bar{P}s_\nu^\mu$ and $\bar{Q}s_\nu^\mu$. It follows that the regular solution of equation (37) is, up to a multiplicative constant,

$$\psi_l(z) = i\left\{\sin\left[\frac{(\mu+\nu)\pi}{2}\right]\bar{P}s_\nu^\mu(z) + \frac{2}{\pi}\cos\left[\frac{(\mu+\nu)\pi}{2}\right]\bar{Q}s_\nu^\mu(z)\right\} \tag{41}$$

The series expansions (38) do not enable us to obtain the asymptotic behavior of $\psi_l(z)$. We must therefore express the regular solution (41) in terms of other linearly independent solutions of the spheroidal wave equation for which the asymptotic behavior is known. Such solutions are (Erdélyi, 1955a) the functions $S_\nu^{\mu(1)}(z,\theta)$ defined by their series expansions in spherical Bessel functions:

$$S_\nu^{\mu(1)}(z,\theta) = (1-z^{-2})^{-\mu/2} s_\nu^\mu(\theta) \sum_{r=-\infty}^{+\infty} a_{\nu,r}^\mu(\theta) j_{\nu+2r}(2\theta^{1/2}z) \tag{42}$$

where the coefficients $a_{\nu,r}^\mu(\theta)$ are the same as in (38), and

$$s_\nu^\mu(\theta) = \left[\sum_{r=-\infty}^{+\infty} (-1)^r a_{\nu,r}^\mu(\theta)\right]^{-1} \tag{43}$$

These solutions have the asymptotic behavior

$$S_\nu^{\mu(1)}(z,\theta) \underset{(2\theta^{1/2}z\to\infty)}{\sim} j_\nu(2\theta^{1/2}z) \sim \frac{1}{2\theta^{1/2}z}\sin(2\theta^{1/2}z - \nu\pi/2) \tag{44}$$

for $|\arg 2\theta^{1/2}z| < \pi$; in our case

$$2\theta^{1/2}z = \omega u \tag{45}$$

so that equation (44) gives the asymptotic behavior for $u \to +\infty$.

The technical problem of expressing the regular solution (41) in terms of the $S_\nu^{\mu(1)}$ is solved in Appendix A. The result is

$$\psi_l(z) = 4e^{-i\mu\pi/2}\frac{\cos\alpha}{\sin 4\alpha}\left[i\frac{e^{-3i\alpha}\cos\alpha}{K_{-\nu-1}^\mu(\theta)} S_{-\nu-1}^{\mu(1)}(z,\theta) + \frac{e^{3i\alpha}\sin\alpha}{K_\nu^\mu(\theta)} S_\nu^{\mu(1)}(z,\theta)\right] \tag{46}$$

with $\alpha = (\mu+\nu)\pi/2$ $(\mu \in \mathbb{Z})$.

The expressions of the complex joining factors $K_\nu^\mu(\theta)$ are complicated and not very illuminating (Erdélyi, 1955a). We only note that, in the case $\theta < 0$, we may define real joining factors $k_\nu^\mu(\theta)$ by

$$K_\nu^\mu(\theta) = e^{3i\pi\nu/2} k_\nu^\mu(\theta) \tag{47}$$

so that equation (46) assumes the real form

$$\psi_l(z) = \frac{4\cos\alpha}{\sin 4\alpha}\left[\frac{\cos\alpha}{k_{-\nu-1}^\mu(\theta)} S_{-\nu-1}^{\mu(1)}(z,\theta) + (-1)^\mu \frac{\sin\alpha}{k_\nu^\mu(\theta)} S_\nu^{\mu(1)}(z,\theta)\right] \tag{48}$$

from which, using (44), the phase shifts δ_l can be extracted.

4. LOW-ENERGY BEHAVIOR

We are primarily interested in the long-wavelength, low-energy domain $\omega\rho_0 \ll 1$, i.e., $|\theta| \ll 1$. In this domain, a solution of equation (39) to first order in θ is

$$\nu = l + \frac{l(l+1)+\mu^2-1}{(l-1/2)(l+1/2)(l+3/2)}\theta + O(\theta^2) \tag{49}$$

Furthermore the joining factors are given, to lowest order in θ, by the formula (Erdélyi, 1955a)

$$\lim_{\theta \to 0} \theta^{-\nu/2} K_\nu^\mu(\theta) = e^{i\nu\pi} \frac{\Gamma(1+\nu-\mu)\Gamma(1/2-\nu)}{2^{\nu+1}\Gamma(\nu+3/2)} \tag{50}$$

so that, for $\mu = 0$, $l \geqslant 0$, or $\mu = 1$, $l \geqslant 1$ [direct inspection of equation (26) shows that there is no scattering in the case $\mu = 1$, $l = 0$]:

$$k_\nu^\mu(\theta) = O(\theta^{l/2}), \qquad k_{-\nu-1}^\mu(\theta) = O(\theta^{-(l+3)/2}) \tag{51}$$

It follows that the ratio of the coefficients of $S_{-\nu-1}^{\mu(1)}$ and $S_\nu^{\mu(1)}$ in the right-hand side of equation (48) is of the order

$$\cot\alpha \frac{k_\nu^\mu(\theta)}{k_{-\nu-1}^\mu(\theta)} = O(\theta^{l+1/2}), \qquad \text{for } \mu + l \text{ even}$$

$$= O(\theta^{l+5/2}), \qquad \text{for } \mu + l \text{ odd} \tag{52}$$

Thus, the dominant contribution to the phase shifts δ_l can be computed [using equations (30), (44), and (49)] from the asymptotic behavior of $S_\nu^{\mu(1)}(z,\theta)$ alone:

$$\delta_l = \frac{\pi\omega^2\rho_0^2}{8} \frac{l(l+1)+\mu^2-1}{(l-1/2)(l+1/2)(l+3/2)} + O(\omega^3\rho_0^3) \tag{53}$$

except for $\mu = l = 0$, in which case equation (48) gives

$$\psi_0(z) = S_0^{0(1)}(z,\theta) - \frac{2}{\pi}\omega\rho_0 S_{-1}^{0(1)}(z,\theta) + O(\omega^2\rho_0^2) \tag{54}$$

from which it follows that

$$\delta_0 = -\frac{2}{\pi}\omega\rho_0 + O(\omega^2\rho_0^2) \qquad (\mu = 0) \tag{55}$$

These results show that the partial-wave amplitudes (31) for the scattering of a scalar wave ($\mu = 0$) by the Ellis geometry vanish in the low-energy limit $\omega \to 0$, except for the s-wave ($l = 0$) amplitude which tends to the finite limit $-a$, where

$$a = \frac{2}{\pi}\rho_0 = \frac{Q^2}{4\pi M} \tag{56}$$

(the classical electromagnetic radius of the extended particle) is the scattering length (Newton, 1966) associated with the repulsive potential

$$U_0(u) = \frac{\rho_0^2}{\left(\rho_0^2 + u^2\right)^2} \tag{57}$$

This result can be checked by noting that the regular solution

$$\phi(\mathbf{x}) = c \arctan\left(\frac{\mu}{\rho_0}\right) \tag{58}$$

of the static equation is asymptotic to

$$\phi_{\text{as}}(\mathbf{x}) = c\left(\frac{\pi}{2} - \frac{\rho_0}{u}\right) = c\phi(\infty)\left(1 - \frac{a}{u}\right) \tag{59}$$

In the case of an electromagnetic wave ($\mu = 1$), the partial-wave amplitudes all vanish in the low-energy limit, and we do *not* recover the Thomson cross section

$$\sigma_T = \frac{32\rho_0^2}{3\pi} \tag{60}$$

The reason for this negative result is simple. In the classical picture of the Compton effect, electromagnetic radiation incident on a point charge imparts to it an oscillatory motion which gives rise to dipole radiation. In our treatment, the geometry being static and spherically symmetric, there can only occur, in the low-energy limit, monopole (s-wave) radiation, which is of course suppressed in the electromagnetic case because of the spin 1 of the photon. To obtain dipole radiation, we should therefore take into account the geometrodynamical breaking of spherical symmetry induced by the energy-momentum density of the incident electromagnetic wave.

APPENDIX A

The regular solution (41) of the spheroidal wave equation is defined in terms of the functions $\overline{P}s_\nu^\mu(z)$ and $\overline{Q}s_\nu^\mu(z)$. To obtain its expression (46) in terms of the functions $S_\nu^{\mu(1)}(z,\theta)$ and $S_{-\nu-1}^{\mu(1)}(z,\theta)$, we proceed in two steps.

The first step is to express the solutions $\bar{P}s_\nu^\mu(z)$ and $\bar{Q}s_\nu^\mu(z)$ in terms of solutions $Ps_\nu^\mu(z)$ and $Qs_\nu^\mu(z)$ defined by the series expansions in (associated) Legendre functions

$$
\begin{aligned}
Ps_\nu^\mu(z) &= \sum_{r=-\infty}^{+\infty} (-1)^r a_{\nu,r}^\mu(\theta) P_{\nu+2r}^\mu(z) \\
Qs_\nu^\mu(z) &= \sum_{r=-\infty}^{+\infty} (-1)^r a_{\nu,r}^\mu(\theta) Q_{\nu+2r}^\mu(z)
\end{aligned}
\tag{A1}
$$

[with the same coefficients $a_{\nu,r}^\mu(\theta)$ as in (38)].

These new solutions are analytic in the complex z plane with the cut $]-\infty, +1]$ (Figure 2). From the relations between (associated) Legendre functions and Legendre functions "on the cut" (Erdélyi, 1955b), we obtain the relations giving the $\bar{P}s$ and $\bar{Q}s$ in terms of the Ps and Qs:

$$
\begin{aligned}
\bar{P}s_\nu^\mu(x) &= \tfrac{1}{2}\left[e^{i\mu\pi/2} Ps_\nu^\mu(x+i0) + e^{-i\mu\pi/2} Ps_\nu^\mu(x-i0)\right] \\
\bar{Q}s_\nu^\mu(x) &= \tfrac{1}{2} e^{-i\mu\pi}\left[e^{-i\mu\pi/2} Qs_\nu^\mu(x+i0) + e^{i\mu\pi/2} Qs_\nu^\mu(x-i0)\right]
\end{aligned}
\tag{A2}
$$

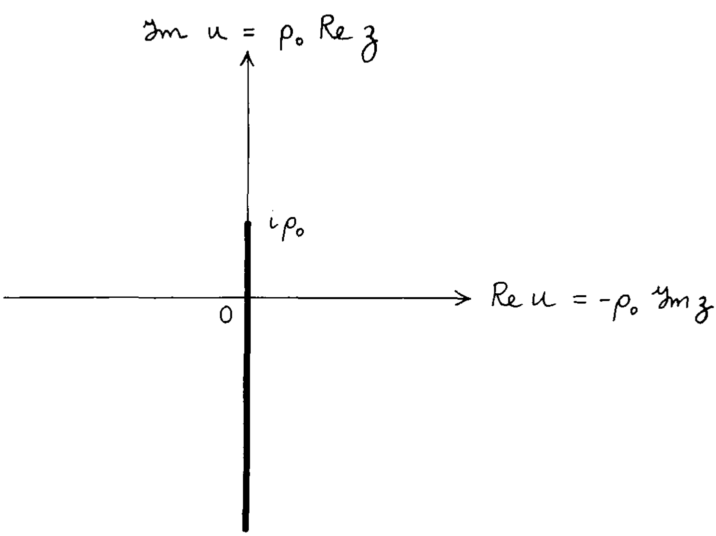

Fig. 2. The domain of analyticity of $Ps_\nu^\mu(z)$ and $Qs_\nu^\mu(z)$.

($x \in]-1, +1[$). Furthermore, the analytic continuation of the Ps and Qs from one side of the cut to the other gives (Meixner, 1951):

$$Ps_\nu^\mu(x+i0) = e^{-i\mu\pi} Ps_\nu^\mu(x-i0)$$
$$Qs_\nu^\mu(x+i0) = e^{i\mu\pi} Qs_\nu^\mu(x-i0) - i\pi e^{i\mu\pi} Ps_\nu^\mu(x-i0) \qquad \text{(A3)}$$

Finally, the transcendental equation giving ν in terms of λ and θ is invariant under the transformation

$$\nu \to -\nu - 1$$

so that there is a relation (Erdélyi, 1955a; Meixner, 1951) giving Ps_ν^μ in terms of the independent solutions Qs_ν^μ and $Qs_{-\nu-1}^\mu$:

$$\pi e^{i\mu\pi} \cos(\nu\pi) Ps_\nu^\mu(z) = \sin[(\mu+\nu)\pi]\, Qs_\nu^\mu(z) + \sin[(\mu-\nu)\pi]\, Qs_{-\nu-1}^\mu(z) \qquad \text{(A4)}$$

Collecting these relations, and analytically continuing the result in the half-plane $\mathrm{Im}\, z < 0$ ($\mathrm{Re}\, u > 0$), gives the regular solution as

$$\psi_l(z) = \frac{2i}{\pi} e^{-i\mu\pi/2} \frac{\cos\alpha}{\cos 2\alpha} \left[e^{-i\alpha} \cos\alpha\, Qs_\nu^\mu(z) + i e^{i\alpha} \sin\alpha\, Qs_{-\nu-1}^\mu(z) \right] \qquad \text{(A5)}$$

(for $\mu \in \mathbb{Z}$), where

$$\alpha = \frac{(\mu+\nu)\pi}{2} \qquad \text{(A6)}$$

The second step involves the identities (Erdélyi, 1955a)

$$S_\nu^{\mu(1)}(z,\theta) = \frac{1}{\pi} e^{-i(\mu+\nu)\pi} \sin[(\mu-\nu)\pi]\, K_\nu^\mu(\theta)\, Qs_{-\nu-1}^\mu(z) \qquad \text{(A7)}$$

where the joining factors $K_\nu^\mu(\theta)$ are given by a ratio of two infinite series in $a_{\nu,r}^\mu(\theta)$.

APPENDIX B

A simple check of formula (53) for the phase shifts may be obtained by using the semiclassical approximation in the domain of large l. Writing the radial equation (15) or (26) as

$$f_l''(u) + \left[\omega^2 - \frac{l'^2}{u^2} + U_{l'}(u) \right] f_l(u) = 0 \qquad \text{(B1)}$$

where

$$l' = l + \tfrac{1}{2} \tag{B2}$$

the large l semiclassical phase shifts are given (Landau and Lifchitz, 1974) by

$$\delta_l \simeq \frac{1}{2}\int_{l'/\omega}^{\infty} \frac{U_{l'}(u)\,du}{\left(\omega^2 - l'^2/u^2\right)^{1/2}} \tag{B3}$$

In the domain of integration,

$$U_{l'}(u) = \frac{l'^2}{u^2} - \frac{l'^2}{\rho_0^2 + u^2} - \frac{(1-\mu^2)\rho_0^2}{\left(\rho_0^2 + u^2\right)^2} \simeq \frac{\rho_0^2\left(l'^2 + \mu^2 - 1\right)}{u^4} \tag{B4}$$

Inserting this approximate effective potential in equation (B3) gives

$$\delta_l \simeq \frac{\pi\omega^2\rho_0^2}{8}\,\frac{l'^2 + \mu^2 - 1}{l'^3} \tag{B5}$$

which agrees with equation (53) for large l.

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